from the three H atoms, or two H's and a D for CH_3D, would be equal to the Van der Waals distance. In the argon matrix the separation of the satellite components is 24 G from the main H lines. The most favorable orientation of the CH_3 group of CH_3D with the three H's coupled equally would require an internuclear separation of approximately 1.3 Å between the atomic H and each of the three CH_3 hydrogens to give such a separation. This does not seem to be an unreasonable figure, although the satellite spacing is larger for the krypton matrix.

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Disturbance of Phonon Distribution by Hot Electrons

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Calculations are made of the steady-state phonon distribution at low temperatures and high electric fields in a many-valley semiconductor, and numerical evaluation is carried out for n-germanium for which all parameters involved are known. It is concluded that the departure from thermal equilibrium will be significant for germanium samples in which the product of carrier concentration n and cross-dimension L is of the order of $10^{13}/cm^2$, and will of course increase as the nL product increases beyond this value. The disturbed phonon distribution is found to be quite anisotropic. The relaxation time tensor and the mobility μ in the presence of the disturbed distribution are calculated. It is found that, when the disturbance is not too large, $\mu \propto (nL)^{-\frac{3}{4}}E^{-\frac{1}{2}}$, where E is electric field intensity. This has been shown to agree with experimental data for n-Ge at 4°K. The question of whether these effects have been observed in p-Ge is discussed.

I. INTRODUCTION

BECAUSE of the high rate of phonon generation by hot carriers, and the long lifetimes for acoustic phonons at low temperatures, it has been reasoned that significant deviations from the thermal equilibrium phonon distribution might be found at low temperatures.[1–4] Experimental evidence that such deviations do in fact occur in n-germanium has been obtained by Zylbersztejn.[4] It is the purpose of the present paper to further explore theoretically the hot-electron-caused disturbance of the phonon distribution in a many-valley semiconductor, and the effects it has on electron transport. Numerical evaluation will be carried out for the case of n-germanium.

II. STEADY-STATE PHONON DISTRIBUTION

Consider the situation at 4°K in an n-germanium sample with saturation carrier concentration of the order of $10^{14}/cm^3$. At low fields there are few free carriers, and no disturbance of the phonon distribution is expected. When the breakdown field is reached, carrier concentration increases rapidly. For a range of fields starting at about two to three times the breakdown field it is found typically,[5] in samples of small cross dimensions, that carrier concentration remains essentially constant, although at a value somewhat less than the saturation value. For fields at the beginning of this range, it is expected that the average energy of the electrons corresponds to a temperature of the order of 30°K since approximately this temperature is required to produce essentially complete thermal ionization of the impurities. At higher fields, when the average energy of the electrons corresponds to a temperature of about 70°K, optical phonon emission will become significant.[6] The calculations of this paper will apply to the range of fields in which carrier concentration is constant and scattering of the electrons is by intravalley acoustic lattice modes and, to a small extent, impurities.

Because the scattering is essentially elastic, whether or not the phonons have the thermal equilibrium distribution, the distribution function $f^{(i)}$ of the carriers in the ith valley will take the form of a function of energy only, $f_0{}^{(i)}(\mathcal{E})$, plus a small drift term, in general not in the

[1] E. M. Conwell, V. J. Fowler, and J. Zucker, quarterly reports, Contract No. DA 36-039-SC-89174, U. S. Army Signal Research and Development Laboratory, Fort Monmouth, New Jersey, 15 May 1962–15 May 1963 (unpublished).
[2] V. V. Paranjape, Proc. Phys. Soc. (London) **80**, 971 (1962).
[3] H. Sato, J. Phys. Soc. Japan **18**, 55 (1963).
[4] A. Zylbersztejn and E. M. Conwell, Phys. Rev. Letters **11**, 417 (1963).
[5] See, for example, S. H. Koenig, R. D. Brown, III, and W. Schillinger, Phys. Rev. **128**, 1668 (1962).
[6] E. M. Conwell, Phys. Chem. Solids **8**, 234 (1959).

field direction because of the anisotropy of the constant energy surfaces. The number of phonons with wave vector $\mathbf{q}$ in the disturbed phonon distribution may also be written as a sum of two terms:

$$N_{\mathbf{q}}=N_{\mathbf{q}0}+q_d g_p(\mathbf{q}). \tag{1}$$

Unlike the situation for the simple model of band structure, the first term, generated by the $f_0^{(i)}(\mathcal{E})$ terms in the electron distribution, is anisotropic because the matrix elements for phonon generation are. It does not, however, represent any net crystal momentum. The second term, generated by the electron drift term, represents a net crystal momentum in a direction, specified by $\mathbf{d}$, that corresponds to the electron drift direction if that is a symmetry direction, but otherwise does not. Since the electron drift term is small in the case under consideration,[7] the phonon drift is expected to be also, and we shall generally neglect it in the considerations that follow. It may be noted that in the field range where impact ionization, which is an inelastic scattering process, is important, the drift terms may be relatively greater in magnitude.

Since the electron-phonon interaction is not strong, we may use perturbation theory to calculate the rates of phonon emission and absorption. Due to the electrons in the ith valley the rate of change of $N_{\mathbf{q}}$ is given by

$$\left(\frac{\partial N_{\mathbf{q}}}{\partial t}\right)^{(i)}=\frac{2\pi}{\hbar}\sum_{\mathbf{k}}\left[|(\mathbf{k},N_{\mathbf{q}}+1|H'|\mathbf{k}+\mathbf{q},N_{\mathbf{q}})|^2\times\delta(\mathcal{E}_{\mathbf{k}}+\hbar\omega_{\mathbf{q}}-\mathcal{E}_{\mathbf{k}+\mathbf{q}})f^{(i)}(\mathbf{k}+\mathbf{q})-|(\mathbf{k}+\mathbf{q},N_{\mathbf{q}}-1|H'|\mathbf{k},N_{\mathbf{q}})|^2\times\delta(\mathcal{E}_{\mathbf{k}+\mathbf{q}}-\hbar\omega_{\mathbf{q}}-\mathcal{E}_{\mathbf{k}})f^{(i)}(\mathbf{k})\right], \tag{2}$$

where the first term gives the rate at which phonons are emitted due to electron transitions from $\mathbf{k}+\mathbf{q}$ to $\mathbf{k}$, the second the rate at which phonons are absorbed in electron transitions from $\mathbf{k}$ to $\mathbf{k}+\mathbf{q}$. The $\mathbf{k}$ vectors are all measured from the valley minimum, and the summation is over all $\mathbf{k}$ vectors in the ith valley. The argument of the δ function is the difference between the final and initial energies of the system.

In converting the summation over $\mathbf{k}$ to an integration, it is convenient to change to the coordinate system in which the constant energy surfaces are spheres rather than ellipsoids. This may be done by the usual substitution

$$k_i=(m_i/m_0)^{1/2}k_i^*, \quad i=x,y,z, \tag{3}$$

where x, y, and z are the principal axes of the ellipsoids, z being the longitudinal axis, the m_i the effective masses in the principal axis directions, and m_0 the free electron mass. In this system the energy of a carrier is $\hbar^2k^{*2}/2m_0$.

[7] For an evaluation of this term see E. M. Conwell, Phys. Chem. Solids (to be published); see also Ref. 3.

A transformation similar to (3) must also be made for other vectors to preserve vector relationships.

Since nothing in the integrand depends on the orientation of $\mathbf{k}^*$ or $\mathbf{k}^*+\mathbf{q}^*$ relative to any fixed axis, we may for convenience choose the z^* axis in the $\mathbf{q}^*$ direction. Then, since the phonon energy may be neglected compared to the hot carrier energy, integration over θ^*, involving only the δ function, gives simply

$$\int_0^{\pi}\delta\left(\frac{\hbar^2}{m_0}k^*q^*\cos\theta^*+\frac{\hbar^2q^{*2}}{2m_0}\right)\sin\theta^*d\theta^*=\frac{m_0}{\hbar^2k^*q^*} \quad \text{for} \quad 0\leq q^*\leq 2k^*. \tag{4}$$

For q^* outside the range specified the integral is zero.

For the many-valley model the matrix elements for scattering by intravalley acoustic modes are of the form[8]

$$|(\mathbf{k}\pm\mathbf{q}|H_\alpha'|\mathbf{k})|^2=\frac{\Xi_\alpha^2\hbar\omega_\alpha(\mathbf{q})}{2V\rho u_\alpha^2(\mathbf{q})}\left\{N_{\mathbf{q}\alpha}+\tfrac{1}{2}+\frac{\delta N_{\mathbf{q}\alpha}}{2}\right\}, \tag{5}$$

where for longitudinal waves $\alpha=l$ and

$$\Xi_l=\Xi_d+\Xi_u\cos^2\theta_i, \tag{5a}$$

while for transverse waves $\alpha=t$ and

$$\Xi_t=\Xi_u\sin\theta_i\cos\theta_i, \tag{5b}$$

Ξ_d and Ξ_u being the deformation potentials for dilatation and uniaxial shear, θ_i the angle between $\mathbf{q}$ and the z axis in the ith valley, ρ the density of the crystal and V its volume, u_α the appropriate sound velocity, and $\delta N_{\mathbf{q}\alpha}=+1$ for emission, -1 for absorption. In what follows we shall neglect the anisotropy of the sound velocity, taking an average value for u_l and for u_t. Also, we shall assume that we have ellipsoids of revolution, with $m_x=m_y=m_t$, $m_z=m_l$. Since we intend to neglect the drift terms, we may replace $N_{\mathbf{q}\alpha}$ in (5) by $N_{\mathbf{q}\alpha 0}$, and $f^{(i)}(\mathbf{k})$ in (2) by $f_0^{(i)}(\mathcal{E})$. After these substitutions nothing in (2) depends on the orientation of $\mathbf{k}^*$, and integration over φ^* yields simply a multiplying factor 2π. Incorporating into (2) the matrix elements (5) and the results of integration over θ^* and φ^*, we obtain

$$\left(\frac{\partial N_{\mathbf{q}\alpha}}{\partial t}\right)^{(i)}=\frac{m_t m_l^{1/2}\Xi_\alpha^2}{2\pi m_0^{1/2}\hbar^2\rho u_\alpha}\frac{q}{q^*}\int_{q^*/2}^{\infty}\{(N_{\mathbf{q}\alpha}+1)f_0^{(i)}(\mathcal{E}+\hbar\omega_q)-N_{\mathbf{q}\alpha}f_0^{(i)}(\mathcal{E})\}k^*dk^*. \tag{6}$$

The subscript 0 on $N_{\mathbf{q}\alpha}$ has been omitted for conciseness of notation.

The form of the distribution function $f_0^{(i)}$ depends on the $N_{\mathbf{q}\alpha}$, on the relative amount of impurity scattering, and on the frequency of interelectron collisions. According to the criterion developed by Frohlich and Paranjape, interelectron collisions are sufficiently fre-

[8] C. Herring and E. Vogt, Phys. Rev. **101**, 944 (1956).

quent at the electron concentrations and energies of interest to have an important influence on the distribution function.[9] We shall therefore assume that the distribution is a Maxwell-Boltzmann one:

$$f_0^{(i)}(\mathcal{E})=[n^{(i)}/N_c(T_e^{(i)})]\exp(-\mathcal{E}/k_0T_e^{(i)}) \quad (7)$$

where

$$N_c(T_e^{(i)})=2(2\pi m_t^{2/3}m_l^{1/3}k_0T_e^{(i)}/h^2)^{3/2}, \quad (8)$$

the effective density of states in the ith valley at a temperature $T_e^{(i)}$. Inserting (7) into (6), performing the integration and summing over the valleys to obtain the total generation rate, we find

$$\frac{\partial N_{q\alpha}}{\partial t}=\frac{m_t(m_lm_0)^{1/2}}{2\pi\hbar^4\rho u_\alpha}\sum_{i=1}^{g}\Xi_\alpha^2k_0T_e^{(i)}\frac{n^{(i)}}{N_c(T_e^{(i)})}\frac{q}{q^*}$$
$$\times\{(N_{q\alpha}+1)e^{-\hbar\omega_{q\alpha}/k_0T_e^{(i)}}-N_{q\alpha}\}e^{-\hbar^2q^{*2}/8m_0k_0T_e^{(i)}}, \quad (9)$$

where g is the number of equivalent valleys. To express all quantities in (9) in the untransformed momentum space, we make use of the relation

$$q^*=q\left(\frac{m_0}{m_t}\right)^{1/2}\left(\sin^2\theta_i+\frac{m_t}{m_l}\cos^2\theta_i\right)^{1/2}$$
$$\equiv q\left(\frac{m_0}{m_t}\right)^{1/2}C^{1/2}(\theta_i). \quad (10)$$

The expression (9) gives the rate of increase of $N_{q\alpha}$ due to electronic processes. To obtain a steady state this rate must be balanced by the rate of decrease due to other processes. From the experimental and theoretical work on phonon drag contributions to the thermoelectric power in germanium,[10,11] it can be deduced that at 4°K, for small disturbances of the phonon distribution, phonon-phonon interactions may be neglected, and boundary scattering is the dominant relaxation mechanism. For that case the rate of relaxation is given by

$$\partial N_{q\alpha}/\partial t=-(N_{q\alpha}-\bar{N}_{q\alpha})/\tau_b, \quad (11)$$

where $\bar{N}_{q\alpha}$ is the thermal equilibrium number of phonons with wave vector $\mathbf{q}$ and polarization α, and τ_b depends on the dimensions of the sample and on whether there is specular reflection at the surface. For the case that we are considering, the disturbance of the phonon distribution is not small, yet not so large that phonon-phonon interactions should again become important. On the other hand, the experimental data of Zylbersztejn[4] show that boundary scattering *is* quite important. For a phonon distribution of the form (1) then, the relaxation rate of the drift term should still be given by τ_b, since this represents a momentum relaxation time. As pointed out by Sato,[3] the relaxation rate of the other term would in general be different since it depends on the rate at which energy rather than momentum is removed from the distribution. To obtain a measure of this rate we assume, as was done by Casimir in the treatment of thermal conductivity,[12] that the scattered radiation leaving a point on the surface has the equilibrium distribution appropriate to the temperature at the point. Since in the usual experimental situation the sample is immersed in liquid helium and the average power input is small even at high fields, the surface temperature must be quite close to the bath temperature. As a consequence, the energy relaxation time must be of the order of the time required for a phonon to reach the surface. Experimental evidence that the energy relaxation time is not much longer than this, at any rate, is provided by the fact that application of flat voltage pulses of several microseconds duration resulted in flat current pulses.[4,13] This indicates that the steady state was achieved in less than a microsecond. For samples with cross dimensions $\frac{1}{2}$ cm or less, the average time required for a phonon to reach the boundary is less than a microsecond. We shall assume therefore that the relaxation rate of the first term of (1) is given by (11) with $\tau_b=L/u_\alpha$, where L is the cross dimension of the sample, assumed small compared to its length. This includes no correction for the fact that some phonons travel a distance smaller than L in getting to the surface, but that seems to be an unnecessary refinement here.

Setting the total rate of change of $N_{q\alpha}$, i.e., the sum of (9) and (11), equal to zero, and solving for $N_{q\alpha}$, we obtain the steady-state value

$$N_{q\alpha}=(\bar{N}_{q\alpha}+\sum_{i=1}^{g}F_\alpha^{(i)}e^{-y^{(i)}C(\theta i)}e^{-\hbar\omega_{q\alpha}/k_0T_e^{(i)}})/$$
$$(1+\sum_{i=1}^{g}F_\alpha^{(i)}e^{-y^{(i)}C(\theta i)}(1-e^{-\hbar\omega_{q\alpha}/k_0T_e^{(i)}})) \quad (12)$$

where

$$F_\alpha^{(i)}=\frac{L}{2}\frac{m_t^{3/2}m_l^{1/2}}{\pi\hbar^4\rho u_\alpha^2}\frac{n^{(i)}}{N_c(T_e^{(i)})}k_0T_e^{(i)}\frac{\Xi_\alpha^2}{C^{1/2}(\theta_i)} \quad (13)$$

$$y^{(i)}=\hbar^2q^2/8m_tk_0T_e^{(i)}. \quad (14)$$

For $\hbar\omega_{q\alpha}\ll k_0T_e^{(i)}$, and $\bar{N}_{q\alpha}$ small compared to the other term in the numerator, both of which conditions are well satisfied for most of the phonons of interest, (12) goes over to the form given earlier[4] (except for a trivial difference in notation). In the form (12) it is easily seen that, for T_e equal to T, the lattice temperature, $N_{q\alpha}=\bar{N}_{q\alpha}$. If we set $g=1$, $m_t=m_l$, $\Xi_u=0$ and $\Xi_d=E_1$, the deformation potential for the simple model, the many-valley model goes over to the simple model, and

[9] H. Frohlich and B. V. Paranjape, Proc. Phys. Soc. (London) **B69**, 21 (1956); see also R. Stratton, Proc. Roy. Soc. (London) **A242**, 355 (1957).
[10] T. H. Geballe and G. W. Hull, Phys. Rev. **94**, 1134 (1954).
[11] C. Herring, Phys. Rev. **96**, 1163 (1954).
[12] H. B. G. Casimir, Physica **5**, 495 (1938).
[13] J. Zucker and D. R. Frankl, General Telephone & Electronics Laboratories Inc. (private communication).

(12) is the same as the expression derived by Paranjape for that case.[2] We note, however, that unless F_α is very large, (12) is not at all in the form for which it can be said that a new phonon temperature exists, as assumed by Sato.[3] It will be seen later that such large values of F_α are not likely to be attained, in Ge at least.

Before we carry out any numerical evaluation of (12), it is useful to consider the size of the phonons with which we are concerned and the values of $\bar{N}_{q\alpha}$. We may obtain an average wave vector $\langle q\rangle$ for these phonons as follows. Because of the condition of conservation of crystal momentum, the average q of the phonons involved in intravalley scattering is of the order of the average k, which we shall take as the k of an electron with the average energy $3k_0T_e/2$. This k is of course a function of direction because of the anisotropy of the constant energy surfaces. If we arbitrarily choose an average mass, to be denoted by $\langle m\rangle$, the average $\mathbf{q}$ may be written

$$\langle q\rangle = (3\langle m\rangle k_0T_e/\hbar^2)^{1/2}. \tag{15}$$

For n-germanium, in which the mass ranges from $m_t = 0.082m_0$ to $m_l = 1.6m_0$, a reasonable average is $0.25m_0$, and we shall use this value of $\langle m\rangle$ to define $\langle q\rangle$. Denoting the energy of the phonon with wave vector $\langle q\rangle$ by $\langle \mathcal{E}_\alpha\rangle$, we find, for longitudinal waves and $T_e = T = 4°$, $\langle \mathcal{E}_l\rangle/k_0T = 0.6$. For a mass of $1.6m_0$ in (15) the phonon energy would be $1.5k_0T$. Thus equipartition is not well satisfied even for thermal electrons at 4°K. At an electron temperature of $10T$ or 40°K, $\langle \mathcal{E}_l\rangle/k_0T = 2$. The Planck formula gives $\langle \bar{N}_{ql}\rangle$ as 0.16 for this phonon energy. These values have been calculated with u_l taken as 5.4×10^5 cm/sec, the value in the [110] direction. Choosing $u_t = 3.2\times10^5$ cm/sec, the average value in the [110] direction, we obtain $\langle \mathcal{E}_t\rangle/k_0T = 1.2$. The thermal equilibrium number of phonons for this energy is 0.43.

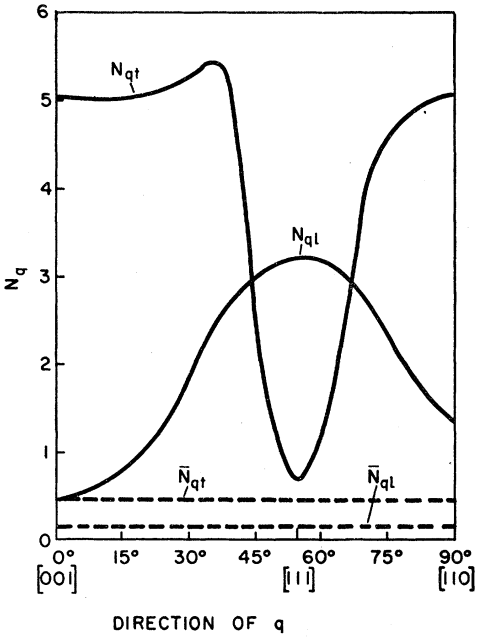

FIG. 1. The solid lines give the steady-state number of longitudinal and transverse phonons in n-Ge as a function of angle from the [001] direction in the [1$\bar{1}$0] plane for an average phonon wave vector defined by Eq. (15) with $\langle m\rangle = 0.25m_0$. Parameters assumed are $T_e = 10\,T = 40°$, $n = 5\times10^{14}$/cm^3, $L = 1$ cm. The dashed lines give the thermal equilibrium number of phonons.

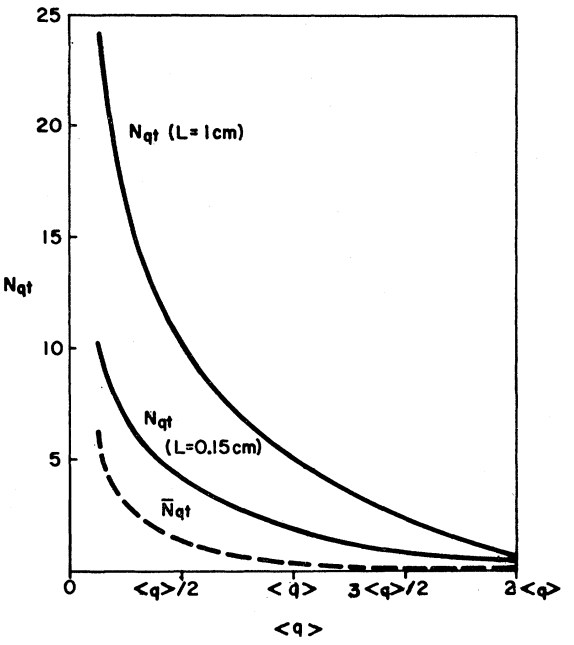

FIG. 2. The solid lines give the steady-state number of transverse phonons as a function of q (in the [100] direction) for n-Ge with $L = 1$ cm, and $L = 0.15$ cm, and the other conditions the same as for Fig. 1. The dashed line gives the thermal equilibrium number of phonons.

Plots of N_{ql} and N_{qt} from the expression (12) for $q = \langle q\rangle$ are given in Fig. 1 as a function of orientation for $T_e = 10\,T = 40°$, $n - 5\times10^{14}$/cm^3, $L = 1$ cm (characteristic of one of the samples in Ref. 4), and the other parameters those of n-germanium. The deformation potentials have been taken as $\Xi_d = -9.07$ eV, and $\Xi_u = 19.3$ eV.[14] It is seen that the departures from equilibrium are considerable. In a sample with $L = 0.15$ cm, a more typical experimental value than 1 cm, the N_{ql} and N_{qt} values would be smaller than those shown in Fig. 1 by only about a factor 2.5, still departing considerably from equilibrium. The anisotropy, as indicated earlier, is what is expected from that of the matrix elements. It is easily seen that the plot of N_{ql} and N_{qt} from 90° to 180° would be the image in the plane at 90° of the plot from 0° to 90°.

It is also instructive to consider the variation of $N_{q\alpha}$ with the magnitude of q. From Eqs. (12)–(14) it can be seen that, if all other factors are constant, as q increases, $N_{q\alpha}$ decreases monotonically. This is shown in Fig. 2, where there is plotted the steady-state number

[14] H. Fritzsche, Phys. Rev. **115**, 336 (1959); S. H. Koenig (private communication).

of transverse phonons as a function of $\mathbf{q}$, for $\mathbf{q}$ in the [100] direction. To show the dependence on L, plots have been made for two values of L. All other parameters are the same as for Fig. 1.

It is interesting to note that although $N_{q\alpha}$ must increase as T_e rises above the lattice temperature, it does not continue to do so indefinitely.[15] In the limit of large enough T_e, provided, of course, that the conditions leading to (12) are still in force, (12) predicts that $N_{q\alpha}$ will decrease as $T_e^{-1/2}$.

III. EFFECTS ON TRANSPORT

The effects of the phonon disturbance on mobility will be considered for the range of electron temperatures, perhaps 35 to 70°, in which n is essentially constant and carrier scattering is intravalley, and by acoustic modes mainly. We shall consider only the case of field applied in the [100] direction, which results in all valleys being at the same temperature T_e. The results are easily generalized to other cases. For this case

$$\frac{n^{(i)}}{N_c(T_e^{(i)})}=\frac{n}{N_c(T_e)}=\frac{n}{N_c(T)}\left(\frac{T}{T_e}\right)^{3/2}, \tag{16}$$

where $N_c(T)$ represents the effective density of states for the entire conduction band at a lattice temperature T. It is useful to introduce a kind of mean free path for acoustic mode scattering, defined by

$$l_1=\frac{\pi\hbar^4\rho u_l^2}{m_t^{3/2}m_l^{1/2}\Xi_d^2 k_0 T}. \tag{17}$$

Under the substitutions that take the many-valley model over to the simple model, l_1 goes over the mean free path for the simple model under equipartition. With (16) and (17) we may rewrite F_α:

$$F_\alpha^{(i)}=\mathfrak{F}_\alpha(\Xi_\alpha/\Xi_d)^2C^{-1/2}(\theta_i), \tag{18}$$

where

$$\mathfrak{F}_l=\frac{L}{2l_1}\frac{n}{N_c(T)}\left(\frac{T}{T_e}\right)^{1/2} \tag{19}$$

and

$$\mathfrak{F}_t=(u_l/u_t)^2\mathfrak{F}_l \tag{20}$$

To obtain the relaxation time tensor, it is necessary to express $N_{q\alpha}$ in terms of the polar angles θ^* and θ'^*, and the azimuthal angles φ^* and φ'^*, of the vectors $\mathbf{k}^*$ and $\mathbf{k}'^*(=\mathbf{k}^*\pm\mathbf{q}^*)$, respectively. Neglecting the small difference between the magnitudes of $\mathbf{k}^*$ and $\mathbf{k}'^*$, we may write

$$N_{q\alpha}=\mathfrak{F}_\alpha\frac{(\bar{N}_{q\alpha}/\mathfrak{F}_\alpha)+\sum_i(\Xi_\alpha/\Xi_d)^2C^{-1/2}(\theta_i)\exp[-f_2x/2]\exp[-(2m_lu_\alpha^2/k_0T_e)^{1/2}f_1x^{1/2}]}{1+\mathfrak{F}_\alpha\sum_i(\Xi_\alpha/\Xi_d)^2C^{-1/2}(\theta_i)\exp[-f_2x/2]\{1-\exp[-(2m_lu_\alpha^2/k_0T_e)^{1/2}f_1x^{1/2}]\}}, \tag{21}$$

where $x=\mathcal{E}/k_0T_e$ and

$$f_1=[(\cos\theta'^*-\cos\theta^*)^2+(m_t/m_l)\{\sin^2\theta'^*+\sin^2\theta^*-2\sin\theta'^*\sin\theta^*\cos(\varphi'^*-\varphi^*)\}]^{1/2} \tag{22}$$

$$f_2=1-\cos\theta'^*\cos\theta^*-\sin\theta'^*\sin\theta^*\cos(\varphi'^*-\varphi^*) \tag{23}$$

The angles θ_i must, of course, also be expressed in terms of θ^*, θ'^*, φ^*, and φ'^*.

For constant energy surfaces that are ellipsoids of revolution, the independent components of the relaxation time tensor may be written[8]

$$\frac{1}{\tau_{\parallel}}=3\pi\int_0^\pi\int_0^\pi\cos\theta^*(\cos\theta'^*-\cos\theta^*)\times\langle\langle\Lambda\rangle\rangle\sin\theta^*d\theta^*\sin\theta'^*d\theta'^*, \tag{24a}$$

$$\frac{1}{\tau_\perp}=\frac{3\pi}{2}\int_0^\pi\int_0^\pi\{\sin^2\theta^*\langle\langle\Lambda\rangle\rangle-\sin\theta'^*\sin\theta^*\langle\cos(\varphi'^*-\varphi^*)\langle\Lambda\rangle\rangle\}\times\sin\theta^*d\theta^*\sin\theta'^*d\theta'^*, \tag{24b}$$

where Λ is the probability per unit time of scattering into unit solid angle in the * space, and the angular brackets indicate azimuthal averages.[8] From the definition

$$\Lambda(\mathbf{k}^*\rightarrow\mathbf{k}'^*)=\frac{m_tm_l^{1/2}V}{2^{3/2}\pi^2\hbar^4}\mathcal{E}^{1/2}|(\mathbf{k}^*\pm\mathbf{q}^*|H_l'+H_t'|\mathbf{k}^*)|^2. \tag{25}$$

Introducing (5) and (17)–(20), (22) and (23), we may rewrite this

$$\Lambda=\frac{1}{2\pi}\frac{u_l}{l_1}\left(\frac{m_l}{m_t}\right)^{1/2}\frac{T_e}{T}\mathfrak{F}_lx\left[\left(\frac{\Xi_l}{\Xi_d}\right)^2f_1\left(\frac{N_{ql}}{\mathfrak{F}_l}+\frac{1}{2\mathfrak{F}_l}\right)+\frac{u_l}{u_t}\frac{m_t}{m_l}\left(\frac{\Xi_t}{\Xi_d}\right)^2f_1\left(\frac{N_{qt}}{\mathfrak{F}_l}+\frac{1}{2\mathfrak{F}_l}\right)\right], \tag{26}$$

where the quantity in brackets contains all the angular dependence. When (26) is inserted into (24a) and (24b), we obtain results that may be written

$$\frac{1}{\tau_\beta}=\frac{u_l}{l_1}\left(\frac{m_l}{m_t}\right)^{1/2}\frac{T_e}{T}\mathfrak{F}_lx\varphi_\beta(x), \tag{27}$$

where β stands for $\parallel$ or $\perp$. The functions $\varphi_\beta(x)$ are quite complicated, and the integration has not been carried out. The form (27) for τ_β is nevertheless quite

[15] Similar behavior is found for the somewhat different situation studied in E. M. Conwell, Phys. Chem. Solids (to be published).

useful when the phonon disturbance is not too large, i.e., for $\mathfrak{F}_l$ not too large, because then, as we shall now show, we may neglect the dependence of $\varphi_\beta(x)$ on $\mathfrak{F}_l$ for the electrons that are important in determining the mobility. For the particular example discussed earlier, when L is in the range 0.1 to 0.15 cm, the term in the denominator of $N_{\mathbf{q}\alpha}$ involving $\mathfrak{F}_l$ is always less than or of the order of 0.25. For electrons of somewhat less than average energy, which are the ones that contribute most to the mobility in this case, the error made in neglecting the term in F_α in the denominator of $N_{\mathbf{q}\alpha}$ is pretty well canceled by neglecting also the term $\bar{N}_{\beta\alpha}$ in the numerator. Since also for these electrons, $\frac{1}{2}$ is small compared to $N_{\mathbf{q}\alpha}$, the terms $(N_{\mathbf{q}\alpha}+\frac{1}{2})/\mathfrak{F}_l$ are essentially independent of $\mathfrak{F}_l$, and therefore $\varphi_\beta(x)$ depends very little on $\mathfrak{F}_l$. To a good approximation, then, for $\mathfrak{F}_l$ not too large (but of course large enough so that $N_{\mathbf{q}\alpha}$ is dominant over $\bar{N}_{\mathbf{q}\alpha}$) $1/\tau_\beta$ is linear in $\mathfrak{F}_l$. For larger values of $\mathfrak{F}_l$, such as obtained in the example given earlier for L about 1 cm, the term involving $\mathfrak{F}_l$ in the denominator of $N_{\mathbf{q}\alpha}$ is of the order of unity, and no simple approximation is possible. For such a case $1/\tau_\beta$ will increase less rapidly than linearly with $\mathfrak{F}_l$.

Since we have assumed a Maxwell-Boltzmann distribution of the carrier energies, the relation between the mobility tensor and the relaxation time tensor is the same as for low fields. In the principal axis system the mobility is a diagonal tensor with components[8]

$$\mu_{\beta\beta}{}^{(i)}=(e/m_\beta)(\langle x\tau_\beta\rangle/\langle x\rangle). \tag{28}$$

For the field directed so that all valleys are at the same temperature, they all make the same contribution to the current. Also, by symmetry, for this direction of field the total current is parallel to the field. It is easily shown then that the over-all mobility is given by the same expression as for low fields[4]:

$$\mu=\frac{e}{3\langle x\rangle}\left[\frac{2\langle x\tau_\perp\rangle}{m_t}+\frac{\langle x\tau_{||}\rangle}{m_l}\right]. \tag{29}$$

Using (27) for $\tau_{||}$ and $\tau_\perp$, and replacing $\mathfrak{F}_l$ with the use of (19), we obtain:

$$\mu=\Phi\frac{e}{(m_tm_l)^{1/2}}\frac{l_1}{u_l}\frac{l_1}{L}\frac{N_c(T)}{n}\left(\frac{T}{T_e}\right)^{1/2}, \tag{30}$$

where Φ is the dimensionless parameter defined by

$$\Phi\equiv\frac{2}{3}\left[\frac{2\langle\varphi_\perp{}^{-1}\rangle}{\langle x\rangle}+\frac{m_t}{m_l}\frac{\langle\varphi_{||}{}^{-1}\rangle}{\langle x\rangle}\right]. \tag{31}$$

According to our earlier considerations, if the phonon disturbance is not too large, Φ depends only weakly on $\mathfrak{F}_l$, and the dependences of μ on n, L and T_e are essentially as given explicitly in (30). This should still be true in the presence of a small amount of impurity scattering, ionized or neutral. The main effects of such scattering would be to modify the value of Φ, and to postpone the region in which (30) is valid to somewhat higher fields.

To determine T_e, we may use the usual method of equating the average rate of energy gain of an electron from the field $e\mu E^2$ to the average rate of its energy loss. If we neglect the small phonon drift, the average rate of loss may be calculated from

$$\left\langle\frac{d\mathcal{E}}{dt}\right\rangle=-\frac{1}{nV}\sum_{\mathbf{q}}\left[\hbar u_{lq}\left(\frac{\partial N_{\mathbf{q}l}}{\partial t}\right)+\hbar u_{tq}\left(\frac{\partial N_{\mathbf{q}t}}{\partial t}\right)\right], \tag{32}$$

with $(\partial N_{\mathbf{q}\alpha}/\partial t)$ given by (9). For $T_e\gg T$, which is the case of present interest, $e^{-\hbar\omega_{q\alpha}/k_0T_e}$ may be replaced by $(1-\hbar\omega_{q\alpha}/k_0T_e)$. The curly bracket in (9) may then be approximated by

$$\{(N_{\mathbf{q}\alpha}+1)e^{-\hbar\omega_{q\alpha}/k_0T_e}-N_{\mathbf{q}\alpha}\}\simeq(1+\sum_i F_\alpha{}^{(i)}e^{-y(i)C(\theta i)\hbar\omega_{q\alpha}/k_0T_e})^{-1}. \tag{33}$$

For $\mathfrak{F}_\alpha$ not too large, the case we are now considering, the term in $\mathfrak{F}_\alpha$ may be neglected. Physically, this indicates that at high enough electron temperature induced emission and absorption cancel each other, and the loss rate is determined by spontaneous emission. When the term in F is neglected, evaluation of (32) is simple. The summation over $\mathbf{q}$ is converted into an integration over q^*, which is easily carried out to give

$$\left\langle\frac{d\mathcal{E}}{dt}\right\rangle=-\frac{8}{\pi^{1/2}}\frac{\Xi_0{}^2}{\Xi_d{}^2}\frac{(2k_0T/m_t)^{1/2}}{l_1}m_tu_l{}^2\left(\frac{T_e}{T}\right)^{3/2} \tag{34}$$

when

$$\Xi_0{}^2=\Xi_d{}^2\left\{\frac{2}{3}+\frac{1}{3}\frac{m_l}{m_t}\left(\frac{\Xi_u}{\Xi_d}+1\right)^2\right\}. \tag{35}$$

It is readily seen that under the usual substitutions (34) goes over to the rate of energy loss for the simple model.[6]

Equating $-\langle d\mathcal{E}/dt\rangle$ of (34) with $e\mu E^2$, we obtain for the electron temperature:

$$\frac{T_e}{T}=\frac{\pi^{1/4}\Phi^{1/2}}{2^{7/4}}\frac{\Xi_d}{\Xi_0}\left(\frac{N_c(T)}{n}\right)^{1/2}\left(\frac{l_1}{L}\right)^{1/2}\times\frac{el_1E}{(m_tm_l{}^{1/2}u_l{}^3)^{1/2}(k_0T)^{1/4}}. \tag{36}$$

It is seen that, when the dependence of Φ on F_l can be neglected, the electron temperature increases linearly with field, and decreases as the square root of n and L. Substituting T_e/T given by (36) into (30), we find

$$\mu=\frac{2^{7/8}\Phi^{3/4}}{\pi^{1/8}}\frac{e}{(m_tm_l)^{1/2}}\frac{l_1}{u_l}\left(\frac{\Xi_0}{\Xi_d}\right)^{1/2}\left(\frac{l_1}{L}\right)^{3/4}\left(\frac{N_c(T)}{n}\right)^{3/4}\times\frac{(m_t{}^{2/3}m_l{}^{1/3}u_l{}^2)^{3/8}(k_0T)^{1/8}}{(el_1E)^{1/2}}. \tag{37}$$

Paranjape[2] has carried out a derivation along similar lines for the simple model of the band structure, with the difference that he has assumed that the phonon lifetime is proportional to $q^{-\gamma}$. This leads to

$$\mu \propto E^{-2(1-\gamma)/(4-\gamma)}. \tag{38}$$

For boundary scattering $\gamma=0$, and (38) gives $\mu \propto E^{-1/2}$, in agreement with (37), or more exactly with μ for the simple model obtained from (37) by the appropriate substitutions. For $\gamma=0$ Paranjape also obtains $\mu \propto (nL)^{-3/4}$, in agreement with (37).

If there were no disturbance of the phonon equilibrium, but the other conditions assumed above were valid, for $T_e \geq 40°$ zero-point lattice conditions would prevail for n-Ge. In that case it can be shown, as was shown earlier for the simple model,[16] that μ would be proportional to $E^{-0.8}$, and independent of n and L, of course.

As discussed earlier,[4] it has been found experimentally at 4° in an n-Ge sample with $n=5\times10^{14}/\text{cm}^3$, $L=0.15$ cm, in the field range from about 35 to 70 V/cm, that the dependence of mobility on field is about as predicted by (37). Comparison of the mobility in this sample with that in a sample of larger L has essentially verified[4] the predicted dependence on L. The situation in p-Ge is less clear. According to 4° measurements reported by Vul and Zavarickaya, on a sample with saturation hole density a little less than $10^{14}/\text{cm}^3$ and cross-dimensions about 0.1 cm×0.7 cm, in the range 20–70 V/cm the mobility is independent of field, while in the range 70–200 V/cm it is proportional to $E^{-1/2}$.[17] In the absence of detailed calculations for p-Ge, it is of course not possible to predict whether a sizeable disturbance of the phonon distribution should exist under these conditions. Certainly, the possibility exists. Since the theoretical prediction of the simple model in the absence of a phonon disturbance is $\mu \propto E^{-0.8}$, Paranjape[4] reasoned that (1) the constant mobility in the range 20–70 V/cm is evidence for a departure from the thermal equilibrium distribution, and (2) $\gamma=1$, since that is the value required, according to Eq. (38), to produce constant mobility. A phonon lifetime $\propto q^{-1}$ may be obtained from phonon-phonon interactions for phonons of transverse branches, although not apparently for longitudinal phonons in Ge.[11] Studies of conductivity as a function of size, similar to those made on n-Ge by Zylbersztejn,[4] are not yet available for p-Ge. However, it is difficult to see why phonon-phonon interactions would be important in p-Ge in this range when they are apparently not in n-Ge. In addition, it must be remembered that, in contrast to the situation for n-Ge, theories of high-field behavior based on the simple model of band structure have not been particularly successful for p-Ge. It has, for example, not been possible to account for the $E^{-0.8}$ dependence of μ observed at 78° for p-Ge.[16] Thus it seems rather more likely that boundary scattering is predominant in p-Ge as well as n-Ge under the experimental conditions of concern, and the failure of the theory to predict constant mobility arises from too simple treatment of transport in the degenerate band structure. In any case, the observation of constant mobility does not now appear to be interpretable as evidence for or against a disturbance of the phonon distribution. Measurements of j versus E for p-Ge samples of different cross-dimensions would be helpful in clarifying the situation.

In conclusion, it is reasonable to expect that disturbances of the phonon distribution similar to that found in n-Ge will be found in other semiconductors at low temperatures.

[16] B. V. Paranjape, Proc. Phys. Soc. (London) **B70**, 628 (1959); see also E. M. Conwell and A. L. Brown, Phys. Chem. Solids **15**, 208 (1960).

[17] B. M. Vul and E. I. Zavarickaya, *Proceedings of the International Conference on Semiconductor Physics, Prague, 1960* (Czechoslovakian Academy of Sciences, Prague, 1961), p. 107.